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STABILITY OF ANNULAR ORTHOTROPIC PLATES

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It is assumed that the orthotropy of the plate material is rectangular or polar, and a uniformly distributed, external compressive or tensile load is applied to the internal boundary of the plate. Stability is analyzed by the Ritz method with the use of Alfutov–Balabukh and Bryan energy criteria. Diagrams of the critical external load and buckling modes as a function of the plate dimensions are given.

Formulation of the Problem. Let a plate be referred to a Cartesian coordinate system so that its middle surface coincides with the x_1Ox_2 coordinate plane. We assume that the plate is made of an orthotropic material and loaded by an in-plane "dead" load q (Fig. 1). Deformed only by in-plane forces, the plate can be in a plane state of equilibrium which is characterized by zero deflections of its middle surface. For a certain level of the external load, a bent equilibrium configuration is possible. This critical value of the external load is generally determined by using the energy stability criteria [1–4]. Indeed, under the requirement of minimum external load, the stability problem of an orthotropic plate is equivalent to the problem of determining the extremals of the Alfutov–Balabukh functional [1–4]:

$$I[w,\varphi,\Phi] = \frac{1}{2}\int\limits_{\Omega} D_{ijkl} w_{,ij} w_{,kl}\,dx_1\,dx_2 + \frac{h}{2}\int\limits_{\Omega} \sigma^{s}_{ij} w_{,i} w_{,j}\,dx_1\,dx_2 - h\int\limits_{\Omega} b_{ijkl}\sigma^{s}_{ij}\sigma^{+}_{kl}\,dx_1\,dx_2, \qquad i,\ j,\ k,\ l = 1,\ 2. \tag{1}$$

Here D_{ijkl} is the flexural-rigidity tensor, b_{ijkl} is the tensor of elastic constants of the plate material, h is the plate thickness, $w(x_1,x_2)$ is a deflection function that must satisfy the kinematic boundary conditions, $\sigma^{s}_{ij} = \delta_{ijkl}\varphi_{,kl}$ are the statically admissible prebuckling stresses, $\sigma^{+}_{ij} = \delta_{ijkl}\Phi_{,kl}$ are the additional stresses that act in the middle surface of the plate and are caused by buckling, $\varphi(x_1,x_2)$ and $\Phi(x_1,x_2)$ are the corresponding stress functions [1–4], and $\delta_{ijkl} = \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}$, δ_{ij}, δ_{kl}, δ_{ik}, and δ_{jl} are the Kronecker symbols. The subscripts after a comma indicate partial differentiation with respect to the corresponding variables.

The variational equation $\delta I = 0$ implies the stability equation and the static boundary conditions for bending and the strain-compatibility equations and the continuity conditions at the contours for the plane and bent configurations. The critical value of functional (1) does not depend on the form of the statically admissible stresses σ^{s}_{ij} if the stresses σ^{+}_{ij} satisfy the equation [2–4]

$$\frac{1}{2}\int\limits_{\Omega} \sigma^{+}_{ij} w_{,i} w_{,j}\,dx_1\,dx_2 - \int\limits_{\Omega} b_{ijkl}\sigma^{+}_{ij}\sigma^{+}_{kl}\,dx_1\,dx_2 = 0. \tag{2}$$

Thus, in solving the stability problems on the basis of functional (1) with Eq. (2) satisfied, there is no need to determine the prebuckling stress state exactly. As this state, one can use any statically admissible state, including the solution of the problem for an isotropic plate. Moreover, if the domain Ω is canonical, the variational problem (1) and (2) can be solved by the Ritz method.

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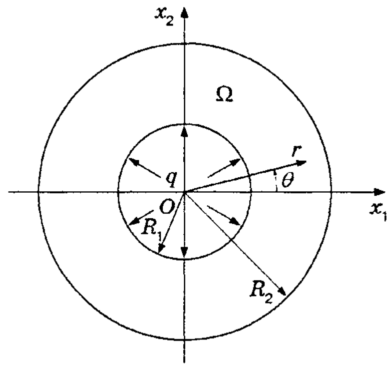

Fig. 1

Stability Analysis of Annular Plates. In polar coordinates (Fig. 1), the dimensionless functional (1) and condition (2) have the form

$$\bar{I} = \frac{1}{2}\int_0^{2\pi}\int_1^{\xi} \bar{D}_{ijkl}v_{ij}v_{kl}\rho\,d\rho\,d\theta + \frac{\lambda}{2}\int_0^{2\pi}\int_1^{\xi} \bar{\sigma}^{c}_{ij}\vartheta_i\vartheta_j\rho\,d\rho\,d\theta - \lambda\int_0^{2\pi}\int_1^{\xi} \bar{b}_{ijkl}\bar{\sigma}^{s}_{ij}\bar{\sigma}^{+}_{kl}\rho\,d\rho\,d\theta (i,\, j,\, k,\, l = 1,\, 2); \quad (3)$$

$$\frac{1}{2}\int_0^{2\pi}\int_1^{\xi} \bar{\sigma}^{+}_{ij}\vartheta_i\vartheta_j\rho\,d\rho\,d\theta - \int_0^{2\pi}\int_1^{\xi} \bar{b}_{ijkl}\bar{\sigma}^{+}_{ij}\bar{\sigma}^{+}_{kl}\rho\,d\rho\,d\theta = 0, \quad i,\, j,\, k,\, l = 1,\, 2. \quad (4)$$

Here

$$v_{11} = \frac{\partial^2 \bar{w}}{\partial \rho^2}, \quad v_{12} = v_{21} = \frac{\partial}{\partial \rho}\Big(\frac{1}{\rho}\frac{\partial \bar{w}}{\partial \theta}\Big), \quad v_{22} = \frac{1}{\rho}\frac{\partial \bar{w}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \bar{w}}{\partial \theta^2}; \quad \vartheta_1 = \frac{\partial \bar{w}}{\partial \rho}, \quad \vartheta_2 = \frac{1}{\rho}\frac{\bar{w}}{\partial \theta};$$

$\rho = r/R_1$ is the dimensionless radius vector, $\xi = R_2/R_1$, $\bar{w} = w/R_1$, $\bar{b}_{ijkl} = b_{ijkl}/\sqrt{b_{1111}b_{2222}}$, $\bar{D}_{ijkl} = D_{ijkl}/\sqrt{D_{1111}D_{2222}}$, $\bar{\sigma}^{+}_{ij} = \sigma^{+}_{ij}\sqrt{b_{1111}b_{2222}}$, $\bar{\sigma}^{s}_{ij} = \sigma^{s}_{ij}/q$, D_{ijkl}, b_{ijkl}, σ^{+}_{ij}, and σ^{s}_{ij} are the components of the corresponding tensors in the orthonormal basis of the polar coordinate system, and $\lambda = qhR_1^2/\sqrt{D_{1111}D_{2222}}$ is the external-load parameter.

We assume that the plate is made of a material with rectangular orthotropy. The stress state of the isotropic plate [5] is used as a statically admissible prebuckling stress state. The stress function $\bar{\Phi}(\rho,\theta)$ is written in the form of a series

$$\bar{\Phi}(\rho,\theta) = A\Big[\rho - (2\xi+1)\Big(\frac{\rho-1}{\xi-1}\Big)^2 + (\xi+1)\Big(\frac{\rho-1}{\xi-1}\Big)^3\Big]\cos\theta$$

$$+ B\Big[\rho - (2\xi+1)\Big(\frac{\rho-1}{\xi-1}\Big)^2 + (\xi+1)\Big(\frac{\rho-1}{\xi-1}\Big)^3\Big]\sin\theta + C\Big[1 - 3\Big(\frac{\rho-1}{\xi-1}\Big)^2 + 2\Big(\frac{\rho-1}{\xi-1}\Big)^3\Big]$$

$$+ \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}(\rho-1)(\rho^{-1}-1)(\rho-\xi)(\rho^{-1}-\xi^{-1})\rho^m[C_{mn}\cos{(n-1)\theta} + D_{mn}\sin n\theta], \quad (5)$$

where A, B, C, C_{mn}, and D_{mn} are arbitrary constants. Series (5) is complete and subject to the boundary conditions [2–4]

$$\bar{\Phi}(\xi,\theta) = 0, \quad \frac{\partial\bar{\Phi}}{\partial\rho}\Big|_{\rho=\xi} = 0, \quad \bar{\sigma}^{+}_{11}(1,\theta) = \bar{\sigma}^{+}_{12}(1,\theta) = 0, \quad \bar{\Phi}(1,\theta) = A\cos\theta + B\sin\theta + C.$$

If the internal contour is clamped and the external contour is traction-free, the deflection function can be written in the form

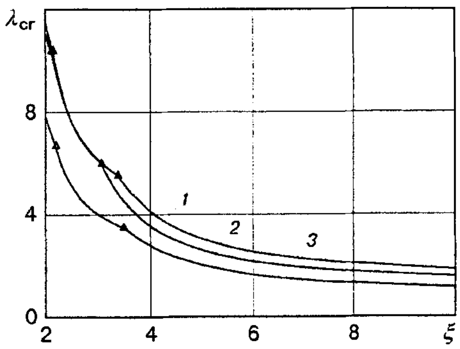

Fig. 2

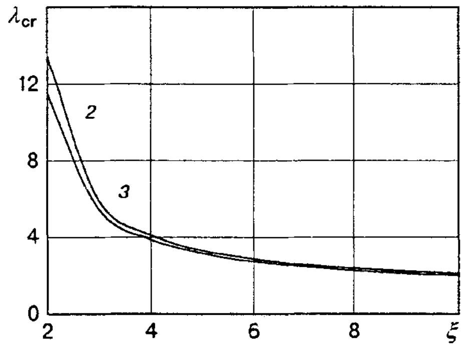

Fig. 3

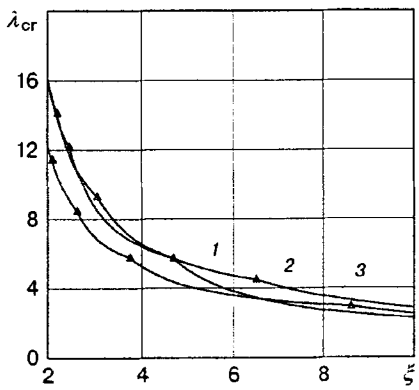

Fig. 4

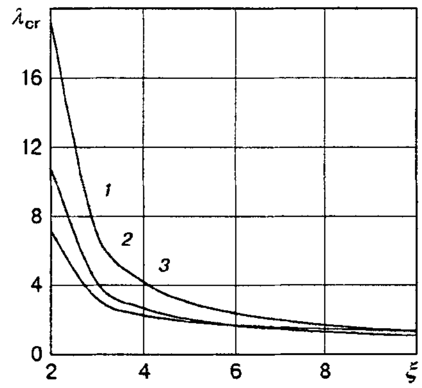

Fig. 5

$$\bar{w}(\rho,\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} (\rho - 1)(\rho^{-1} - 1)\rho^{m}[A_{mn}\cos{(n-1)\theta} + B_{mn}\sin{n\theta}]. \tag{6}$$

Substitution of series (5) and (6) into functional (3) and integral condition (4) and evaluation of the corresponding integrals reduce the stability problem to a problem of determining the conditional extremum of the function $\bar{I}[A, B, C, A_{mn}, B_{mn}, C_{mn}, D_{mn}]$, which was solved by the Lagrange multiplier method [2–4]. The stationarity conditions lead to the generalized eigenvalue problem for the load parameter. The minimum positive and maximum negative values from the eigenvalue spectrum correspond to the critical load λ_{cr} for compression ($q > 0$) and that for tension ($q < 0$), respectively.

For a plate made of a material with a constant cylindrical orthotropy, the exact prebuckling stresses can be found [5]. Therefore, it is expedient to seek the solution of the stability problem with the use of the Bryan functional [1–4]

$$\bar{I}_1 = \frac{1}{2}\int_0^{2\pi}\int_1^{\xi} \bar{D}_{ijkl}v_{ij}v_{kl}\rho\, d\rho\, d\theta + \frac{\lambda}{2}\int_0^{2\pi}\int_1^{\xi} \bar{\sigma}^0_{ij}\vartheta_i\vartheta_j\rho\, d\rho\, d\theta, \quad i,\ j,\ k,\ l = 1,\ 2. \tag{7}$$

The deflection of a plate whose external contour is clamped and whose internal contour is traction-free can be written in the form

$$\bar{w}(\rho,\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn}(\rho - \xi)(\rho^{-1} - \xi^{-1})\rho^{m}\cos{(n-1)\theta}. \tag{8}$$

Substituting series (8) into functional (7) and evaluating the corresponding integrals, we obtain a problem of determining the absolute extremum of the function $\bar{I}_1[A_{mn}]$. The stationarity conditions $\partial\bar{I}_1/\partial A_{mn} = 0$ lead to a homogeneous system of equations linear in the parameters A_{mn}. The minimum positive value of the external load parameter λ_{cr} for which the system has a nontrivial solution is critical.

Some Results. In numerical implementation of the method described above, as many terms (less than 40) were retained in the series for the deflection and stress functions as was necessary to determine the critical load with accuracy to three significant digits. The plates made of glass-cloth-base laminate, glass-reinforced plastic, and boron plastic were calculated. In Figs. 2–5, curve 1 refers to the glass-cloth-base laminate with $E_1 = 2.15 \cdot 10^4$ MPa, $E_2 = 1.98 \cdot 10^4$ MPa, $G = 0.401 \cdot 10^4$ MPa, $\nu_{12} = 0.14$, and $\nu_{21} = 0.152$, curve 2 to glass-reinforced plastic with $E_1 = 6.25 \cdot 10^4$ MPa, $E_2 = 2.12 \cdot 10^4$ MPa, $G = 0.90 \cdot 10^4$ MPa, $\nu_{12} = 0.073$, and $\nu_{21} = 0.215$, and curve 3 to boron plastic with $E_1 = 21.1 \cdot 10^4$ MPa, and $E_2 = 2.11 \cdot 10^4$ MPa, $G = 0.85 \cdot 10^4$ MPa, $\nu_{12} = 0.35$, and $\nu_{21} = 0.035$.

Figures 2 and 3 show the critical load versus the ratio of the radii for a plate of rectangular orthotropy whose internal edge is clamped and whose external edge is traction-free. Figures 2 and 3 correspond to the loads $q < 0$ (tension) and $q > 0$ (compression), respectively. Similar results, which were obtained under the assumption that the orthotropy of the above-indicated materials is cylindrical (the internal edge of the plate is free and the external edge is clamped), are shown in Figs. 4 and 5 (for $q < 0$ and $q > 0$, respectively). In the case of tensile buckling $q < 0$, the number of nodal diameters of the buckling mode changes as ξ varies. At the points where the transition from one mode to another occurs, the curves have breaks (points in Figs. 2 and 4). Examples of these modes are shown in Fig. 6 [for $\xi = 3$ (a) and 4 (b), respectively] for an annular orthotropic boron-plastic plate with a free external edge and a clamped internal edge. For compressive loads, the buckling modes have no nodal diameters; moreover, the cylindrically orthotropic plates buckle axisymmetrically.

The results show the efficiency of the energy criterion proposed. This approach can be extended to other loading cases and supporting conditions and generalized to the case of the heterogeneous elastic characteristics of the material.

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